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HAUSDORFF MEASURES ON JULIA SETS OF SUBEXPANDING RATIONAL MAPS

BY

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ABSTRACT

Let h be the Hausdorff dimension of the Julia set $J(R)$ of a Misiurewicz's rational map $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ (subexpanding case). We prove that the h-dimensional Hausdorff measure H_h on $J(R)$ is finite, positive and the only h-conformal measure for $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ up to a multiplicative constant. Moreover, we show that there exists a unique R-invariant measure on $J(R)$ equivalent to H_h.

1. Introduction

Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree ≥ 2 on the Riemann sphere equipped with the spherical metric. Denote by $J(R)$ the Julia set of R. It is well known that the Julia set $J(R)$ is non-empty, perfect and fully invariant, which means that

$$R^{-1}(J(R)) = J(R) = R(J(R)).$$

Moreover, $J(R)$ is the closure of periodic sources of R and $R|_{J(R)}$ is topologically exact. For the definitions and basic properties of rational maps and their Julia sets we refer to Brolin ([3]), Blanchard ([2]) and Devaney ([4]).

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In this paper, like in [1], [5], [6], [7], [8] and [16], we investigate relations between Hausdorff and conformal measures on Julia sets.

The definition of conformal measures for rational maps was first given by Sullivan (see [14]) as a modification of the Patterson measure for limit sets of Fuchsian groups (see [12]). Let $t > 0$. A probability measure m on $J(R)$ is called t-conformal for $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ if

$$m(R(A)) = \int_A |R'|^t \, dm$$

for every Borel set $A \subset J(R)$ such that $R|_A$ is injective. A more general definition, showing the connection to ergodic theory, has been given by us earlier (see [7]). It follows from topological exactness of $R|_{J(R)}$ that a conformal measure m is positive on non-empty open sets and therefore

$$M(r) = \inf\{m(B(z,r)) : z \in J(R)\} > 0 \tag{1.1}$$

for every $r > 0$.

Let (X, ρ) be a metric space, let $t \geq 0$ and let A be a subset of X. The outer t-dimensional Hausdorff measure of a set $A \subset X$ is defined by

$$\mathrm{H}_t(A) = \lim_{r \to 0} \left(\inf \sum_{j=0}^{\infty} (\mathrm{diam}(U_j))^t \right)$$

where the infinium is taken over all countable covers $\{U_j\}$ of A by balls in X of radii not exceeding r. The Hausdorff dimension $\mathrm{HD}(A)$ of A is defined to be the infimum of all s so that $\mathrm{H}_s(A) = 0$.

Denote by $N(A, r)$ the minimum number of balls in X with radii not exceeding r which are needed to cover A. The lower and upper box dimensions of A are defined by

$$\underline{\mathrm{bD}}(A) = \liminf_{r \to 0} \frac{\log N(A,r)}{-\log r} \quad \text{and} \quad \overline{\mathrm{bD}}(A) = \limsup_{r \to 0} \frac{\log N(A,r)}{-\log r}$$

respectively. If $\underline{\mathrm{bD}}(A) = \overline{\mathrm{bD}}(A)$, this common value is denoted by $\mathrm{bD}(A)$ and called the box dimension of A.

It is easy to check that any positive and finite t-dimensional Hausdorff measure on the Julia set $J(R)$ is t-conformal (after normalization of course) but not necessarily conversely and in fact the relations between Hausdorff and conformal measures and between Hausdorff and box dimensions of $J(R)$ depend heavily on the type of rational map under consideration. In the expanding case all basic problems are solved, mostly due to Bowen and Sullivan. Let $h = \mathrm{H}D(J(R))$ denote the Hausdorff dimension of the Julia set of an expanding rational map R. Then the h-dimensional Hausdorff measure H_h on $J(R)$ is positive and finite, and (after normalization to $\mathrm{H}_h(J(R)) = 1$) it is the only h-conformal measure. Moreover, there are no t-conformal measures with $t \neq h$, and there exists a unique R-invariant probability measure μ on $J(R)$ equivalent to H_h. μ is the unique equilibrium state for the function $-h \log |R'|$ and satisfies $h_\mu(R) - h \int \log |R'| \, d\mu = 0$. Hausdorff and box dimensions are also equal ($h = \mathrm{bD}(J(R))$).

In [1], [5], [6] and [8] we have dealt with parabolic (expansive but not expanding) rational maps. For $t \geq h = \mathrm{H}D(J(R))$ there exists a t-conformal measure, always purely atomic if $t \neq h$, and there are no t-conformal measures for $t < h$. There exists exactly one h-conformal measure which is nonatomic, equal to the normalized h-dimensional Hausdorff measure for $h \geq 1$ and equal to the h-dimensional packing measure for $h \leq 1$. Moreover, there exists an invariant measure equivalent to the h-conformal measure, either finite or σ-finite.

In the general case of a rational map (when the Julia set contains a critical point) we dealt with the question of determining h by the existence of conformal measures in [7]. In some special cases (including subexpanding rational maps) we showed that there are t-conformal measures satisfying $h \geq t$.

In this paper we study a special class of rational maps for which the Julia sets contain critical points, but, on the other hand, which behave very much like expanding maps. These maps are defined as follows: A rational map $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is called a Misiurewicz's rational map (or subexpanding) if the restriction of R to the intersection of the Julia set $J(R)$ and the ω-limit set of critical points of R is expanding with respect to the spherical metric on $J(R)$. This definition can be found in [10] and is analogous to a corresponding definition for maps of the interval by Misiurewicz. We shall prove that there exists exactly one h-conformal measure m and m is equivalent to the h-dimensional Hausdorff measure on $J(R)$ with constant Radon–Nikodym derivative. Moreover, there exists a unique, ergodic, R-invariant probability measure equivalent to m, and the Hausdorff and box dimensions are equal. All these results are contained in Section 4.

The strategy of our approach is the following. First, basically repeating arguments from [16] and [5], we show that if m is a t-conformal measure then $\mathrm{H}_t \ll m$ with bounded Radon–Nikodym derivative. The important step in the present proof is the following (essentially Lemma 3.7 in Section 3): If m is a nonatomic h-conformal measure then $m \ll \mathrm{H}_h$ with bounded Radon–Nikodym derivative. It is this fact which makes the subexpanding case similar to the expanding one, and which is not true, for example, in the parabolic case. Using the construction of conformal measures in [7] we then prove the existence of a nonatomic h-conformal measure. These informations are sufficient to conclude that $0 < \mathrm{H}_h(J(R)) < \infty$ and $h = \mathrm{bD}(J(R))$.

The existence and uniqueness of an R-invariant probability measure equivalent to H_h has been proved in [10] under the assumption that $J(R) = \overline{\mathbb{C}}$. In this case $h = 2$ and H_h is up to a multiplicative constant the Lebesgue measure on $\overline{\mathbb{C}}$. Since the same proof works in our general case we obtain the existence and uniqueness of an R-invariant equivalent probability measure and an immediate consequence is the uniqueness of the h-conformal measure.

All distances and derivatives appearing in this paper are considered with respect to the spherical metric on $\overline{\mathbb{C}}$. However, to keep our exposition more readable we use $|z - y|$ for the distance between y and z. A ball of (spherical) radius r around $x \in \overline{\mathbb{C}}$ (resp. $A \subset \overline{\mathbb{C}}$) will be denoted by $B(x,r)$ (resp. $B(A,r)$).

The following version of the Köbe Distortion Theorem (see [11], comp. [16]) will be used several times.

KÖBE DISTORTION THEOREM (KDT): *Let $\varepsilon > 0$. Then there exists a function $k_\varepsilon : [0,1) \to [1,\infty)$ such that for any $y,z \in \overline{\mathbb{C}}$, $r > 0$, $t \in [0,1)$ and any univalent analytic function $H : B(z,r) \to \overline{\mathbb{C}} \setminus B(y,\varepsilon)$ we have*

$$\sup\{|H'(x)| : x \in B(z,tr)\} \le k_\varepsilon(t) \inf\{|H'(x)| : x \in B(z,tr)\}.$$

For brevity we refer to this theorem as KDT, we put $k = k_{1/2}$ and $K = k_{1/2}(1/2)$. In later applications we only consider families of holomorphic inverse branches of positive iterates of rational functions defined on some open balls of $\overline{\mathbb{C}}$. Although it is not indicated later on, the radii of these balls are always assumed to be so small that the following conditions are satisfied.

First, one assumes the radii to be so small that at least three different periodic cycles lie outside of the ball considered. Then, if necessary, one uses Montel's theorem and passes to a ball of suitably smaller radius so that the assumptions of KDT are satisfied with $\varepsilon = 1/2$ for all inverse branches in the family under consideration.

2. Preliminary results

Let $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Denote by $\mathrm{Crit}(R)$ the set of critical points of R and by $CV(R) = R(\mathrm{Crit}(R))$ the set of critical values of R. In this section we collect a few simple lemmas which will be needed in Section 3.

LEMMA 2.1: *For every $\varepsilon > 0$ there exist $0 < \sigma(\varepsilon) < \varepsilon$ and $\beta(\varepsilon) > 0$ such that for every $z \notin B(\mathrm{Crit}(R), \varepsilon)$ the restriction $R|_{B(z,\sigma(\varepsilon))}$ is injective and there exists a unique holomorphic inverse branch $R_z^{-1} : B(R(z), \beta(\varepsilon)) \to \overline{\mathbb{C}}$ satisfying $R_z^{-1}(R(z)) = z$.*

The proof of this lemma is evident. The existence of $\sigma(\varepsilon)$ is completely obvious. For $\beta(\varepsilon)$ one can take $\varepsilon \inf\{|R'(x)| : x \notin B(\mathrm{Crit}(R), \varepsilon - \sigma(\varepsilon))\}$.

The next lemma can be found in [10], for example. For completeness we repeat here its short and simple proof.

LEMMA 2.2: *$\forall \varepsilon > 0\ \forall \lambda > 1\ \exists \theta = \theta(\varepsilon, \lambda) \leq \min\{\sigma(\varepsilon), \beta(\varepsilon)\}$ such that the following holds: If $n \geq 1$ is an integer and $z \in \overline{\mathbb{C}}$ such that*

$$\{z, R(z), \ldots, R^{n-1}(z)\} \subset \overline{\mathbb{C}} \setminus B(\mathrm{Crit}(R), \varepsilon)$$

and $|R'(R^j(z))| \geq \lambda$ for every $j = 0, 1, \ldots, n-1$, then there exists a unique holomorphic inverse branch $R_z^{-n} : B(R^n(z), 2\theta) \to \overline{\mathbb{C}}$ such that $R_z^{-n}(R^n(z)) = z$.

Moreover, for any $\delta \leq 2\theta$

$$R_z^{-n}(B(R^n(z), \delta)) \subset B(z, \delta).$$

Proof: Take $t \in (0, 1)$ so small that $\lambda^{-1}k(t) < 1$, where $k(t)$ is the function given in KDT. It follows from Lemma 2.1 that for every $j = 1, 2, \ldots, n$ there exists a unique holomorphic inverse branch $R_j^{-1} : B(R^j(z), \beta(\varepsilon)) \to \overline{\mathbb{C}}$ satisfying $R_j^{-1}(R^j(z)) = R^{j-1}(z)$. Hence, in view of KDT,

$$R_j^{-1}(B(R^j(z), t\beta(\varepsilon))) \subset B(R^{j-1}(z), \lambda^{-1}k(t)t\beta(\varepsilon)) \subset B(R^{j-1}(z), t\beta(\varepsilon))$$

and therefore the composition

$$R_1^{-1} \circ R_2^{-1} \circ \cdots \circ R_n^{-1}$$

is well defined on $B(R^n(z), t\beta(\varepsilon))$ and satisfies $R_1^{-1} \circ R_2^{-1} \circ \cdots \circ R_n^{-1}(R^n(z)) = z$. Putting

$$\theta = \frac{1}{2} t\beta(\varepsilon) \quad \text{and} \quad R_z^{-n} = R_1^{-1} \circ R_2^{-1} \circ \cdots \circ R_n^{-1}$$

the proof is finished. ∎

The next lemma is also quite obvious and stated without proof.

LEMMA 2.3: *Let (X, ρ) be a metric space and let ν be a Borel probability measure on X. Fix $x \in X$ and $\alpha > 0$. Assume that there exist $C_1, C_2 \geq 1$ and $r_0 > 0$ such that for all $0 < r < r_0$*

$$\exists r_1 \leq C_1 r \ \exists r_2 \geq C_1^{-1} r \text{ so that}$$

$$\nu(B(x, r_1)) \geq C_2^{-1} r^\alpha \quad \text{and} \quad \nu(B(x, r_2)) \leq C_2 r^\alpha. \tag{2.1}$$

Then $\exists C_3 \geq 1 \ \forall 0 < r \leq 1$

$$C_3^{-1} \leq \frac{\nu(B(x, r))}{r^\alpha} \leq C_3.$$

LEMMA 2.4: *Let m be a t-conformal measure on $J(R)$ and let $z \in J(R)$. Let $\xi > 0$ be so small that for every point $x \in \overline{\mathbb{C}}$ there exist at least three different periodic points $a, b, c \in \overline{\mathbb{C}}$ whose forward trajectories do not intersect the ball $B(x, \xi)$.*

(a) If there exists a sequence $n_k = n_k(z) \to \infty$ and $0 < \xi_1(z) \leq \xi$ such that for every $k \geq 1$ there exists a holomorphic inverse branch $R_z^{-n_k} : B(R^{n_k}(z), \xi_1(z)) \to \overline{\mathbb{C}}$ such that $R_z^{-n_k}(R^{n_k}(z)) = z$, then

$$\limsup_{k \to \infty} |(R^{n_k})'(z)| = \infty. \tag{2.2}$$

Moreover, there exist a constant $C_4 \geq 1$ (depending on $\xi_1(z)$) and a sequence $\{r_k(z)\}_{k=1}^\infty$ of positive integers such that $\lim_{k\to\infty} r_k(z) = 0$ and

$$C_4^{-1} \leq \frac{m(B(z, r_k(z)))}{r_k^t(z)} \leq C_4.$$

(b) Suppose that there exist constants $C_1, C_2, C_5 \geq 1$ (depending on z) and $0 < \xi_2(z) \leq \xi$ such that:

If $0 < r < \xi_2(z)$, then (2.1) holds with $\nu = m$ or there exists $n \geq 1$ such that

$$C_5^{-1} \leq r|(R^n)'(z)| \leq C_5 \tag{2.3}$$

and such that there exists a unique holomorphic inverse branch

$$R_z^{-n} : B(R^n(z), \xi_2(z)) \to \overline{\mathbb{C}} \tag{2.4}$$

satisfying $R_z^{-n}(R^n(z)) = z$.
Then there exists a constant $C_6 \geq 1$ (depending on C_1, C_2, C_5 and $\xi_2(z)$) such that for every $0 < r \leq 1$

$$C_6^{-1} \leq \frac{m(B(z,r))}{r^t(z)} \leq C_6.$$

Remark: Lemma 2.4(b) is the essential assertion used later on. The formulation here is somewhat complicated, but it will be used exactly in this form in Lemma 3.7 below, when we show that for each z and r at least one of the two assumptions is satisfied. Certainly, (2.3) and (2.4) imply (2.1), but we need a uniform constant C_6 and therefore have chosen this formulation. ∎

Proof: (a) Passing to a subsequence one can suppose that for some $y \in J(R)$

$$\lim_{k\to\infty} R^{n_k}(z) = y,$$

and for every $k \geq 1$

$$|y - R^{n_k}(z)| \leq \tfrac{1}{2}\gamma, \tag{2.5}$$

where $\gamma = \frac{1}{2}\xi_1(z)$. In view of this $B(y,\gamma) \subset B(R^{n_k}(z), 2\gamma)$ and hence the family $\{R_z^{-n_k} : B(y,\gamma) \to \overline{\mathbb{C}}\}_{k=1}^{\infty}$ is well defined. Since $\gamma \leq \xi$, it follows from Montel's theorem that this family is normal. Since $y \in J(T)$, all the accumulation points of this family are constant functions. In particular,

$$\limsup_{k\to\infty} |(R^{-n_k})'(R^{n_k}(z))| = 0.$$

So, formula (2.2) is proved.

In order to prove the other part of (a), take $\zeta = \gamma/K$, where $K = k(1/2)$ is the constant defined in KDT, and put

$$r_k = r_k(z) = |(R^{-n_k})'(R^{n_k}(z))|\zeta.$$

By the choice of ζ, applying KDT to the map $R_z^{-n_k} : B(R^{n_k}(z), 2\gamma) \to \overline{\mathbb{C}}$, we get

$$B(z, r_k) \subset R_z^{-n_k}(B(R^{n_k}(z), K\zeta))$$

and

$$B(z, r_k) \supset R_z^{-n_k}(B(R^{n_k}(z), K^{-1}\zeta)).$$

Therefore, using (1.1), KDT, and conformality of the measure m, we obtain

$$\begin{aligned} m(B(z, r_k)) &\geq K^{-t}|(R^{-n_k})'(R^{n_k}(z))|^t m(B(R^{n_k}(z), K^{-1}\zeta)) \\ &\geq K^{-t}\zeta^{-t} r_k^t M(K^{-1}\zeta) = M(K^{-1}\zeta)(K\zeta)^{-t} r_k^t. \end{aligned}$$

Similarly $m(B(z, r_k)) \leq (K\zeta^{-1})^t r_k^t$. The proof of part (a) is finished.

(b) By Lemma 2.3 and the assumption it is sufficient to show that (2.1) holds for all r satisfying (2.3) and (2.4). But this follows immediately as in part (a) from KDT. ∎

3. Volume Lemmas for Conformal Measures

Recall that the ω-limit set of the set $CV(R)$ of critical values of $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is defined by

$$\Omega(R) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{k=n}^{\infty} R^k(CV(R))}.$$

In other words $z \in \Omega(R)$ if and only if there exist $c \in CV(R)$ and a sequence $n_k \uparrow \infty$ $(k \geq 1)$ of positive integers such that $z = \lim_{k\to\infty} R^{n_k}(c)$.

Put

$$\omega(R) = \Omega(R) \cap J(R).$$

From now on we assume that R is a Misiurewicz's (subexpanding) mapping, that is, that $R|_{\omega(R)}$ is expanding which means that

$$\exists s \geq 1 \ \exists \lambda' > 1 \ \forall z \in \omega(R) \quad |(R^s)'(z)| \geq \lambda'.$$

Put $T = R^s$. Since $\omega(T) \subset \omega(R)$, we also have that for every $z \in \omega(T)$

$$|T'(z)| \geq \lambda'.$$

Hence T is also a Misiurewicz's map and, in particular, $\mathrm{Crit}(T) \cap \omega(T) = \emptyset$. Since T' is continuous, there exist $\eta > 0$ and $1 < \lambda \leq \lambda'$ such that

$$\forall z \in B(\omega(T), 2\eta) \quad |T'(z)| \geq \lambda. \tag{3.1}$$

By definition of $\omega(T)$ there exists $p \geq 1$ such that

$$\bigcup_{n=p-1}^{\infty} T^n(CV(T)) \subset B(\omega(T), \eta). \tag{3.2}$$

Therefore, setting

$$\|(T^n)'\| = \sup\{|(T^n)'(z)| : z \in \overline{\mathbb{C}}\} \qquad \text{for } n \geq 1$$

and $\gamma = \eta(2\|(T^{p-1})'\|)^{-1}$, we obtain the following.

LEMMA 3.1: *If $z \in J(T)$, $n \geq p-1$ and $T^n(z) \notin B(\omega(T), 2\eta)$, then $T^{n-p+1}(z) \notin B(\bigcup_{j=0}^{\infty} T^j(CV(T)), 2\gamma)$ and hence there exists a unique holomorphic inverse branch*

$$T_z^{-n+p-1} : B(T^{n-p+1}(z), 2\gamma) \to \overline{\mathbb{C}}$$

such that $T_z^{-n+p-1}(T^{n-p+1}(z)) = z$.

Fix $0 < \varepsilon < \gamma/3$ so small that the following four conditions (3.3)–(3.6) are satisfied:

$$B(\mathrm{Crit}(T), \varepsilon) \cap B(\omega(T), 2\eta) = \emptyset, \tag{3.3}$$

there exists $A \geq 1$ such that for every $c \in \mathrm{Crit}(T^p)$ and every $z \in B(c, 2\varepsilon)$

$$A^{-1}|z-c|^q \leq |T^p(z) - T^p(c)| \leq A|z-c|^q \tag{3.4}$$

and

$$A^{-1}|z-c|^{q-1} \leq |(T^p)'(z)| \leq A|z-c|^{q-1} \tag{3.5}$$

where $q = q(c)$ is the order of T^p at the critical point c, and finally

$$B(c_1, 2\varepsilon) \cap B(c_2, 2\varepsilon) = \emptyset \tag{3.6}$$

for every two different critical points c_1, c_2 of T^p.

Define constants σ, β and θ as in Section 2 for ε and the mapping T. These constants are fixed throughout the remaining part of the paper.

Immediately from (3.6) we obtain the following.

LEMMA 3.2: *There exists $0 < \tau < 1$ such that for every critical point c of T^p and every $z \in B(c, \varepsilon)$ there exists a unique holomorphic inverse branch T_z^{-p} : $B(T^p(z), \tau|T^p(z) - T^p(c)|) \to B(z, \gamma)$ such that $T_z^{-p}(T^p(z)) = z$.*

Let

$$0 < \delta \le \min\{\theta, \eta, \tfrac{1}{3}\gamma\tau K^{-1}A^{-2}\} \tag{3.7}$$

be so small that if $z \notin B(\mathrm{Crit}(T^p), \varepsilon)$ then there exists a unique holomorphic inverse branch $T_z^{-p} : B(T^p(z), 2\delta) \to \overline{\mathbb{C}}$ such that $T_z^{-p}(T^p(z)) = z$ and

$$\mathrm{diam}(T_z^{-p}(B(T^p(z), 2\delta))) < \gamma. \tag{3.8}$$

Let

$$S = \bigcup_{n=0}^{\infty} T^{-n}(\mathrm{Crit}(T)).$$

Next we prove the following result which has some preliminary geometric consequences.

LEMMA 3.3: *If $z \in J(T) \setminus S$ then*

$$\limsup_{n\to\infty} |(T^n)'(z)| = \infty. \tag{3.9}$$

Moreover, there exists a sequence $\{r_k(z)\}_{k=1}^{\infty}$ of positive integers such that $\lim_{k\to\infty} r_k(z) = 0$ and such that the following holds:
If m is a t-conformal measure for $T : J(T) \to J(T)$, then there exists a constant $C_7 \ge 1$ (not depending on z) such that for every $k \ge 1$

$$C_7^{-1} \le \frac{m(B(z, r_k(z)))}{r_k^t(z)} \le C_7. \tag{3.10}$$

Proof: We only need to check that the assumptions of Lemma 2.4(a) are satisfied for the point z and that $\inf\{\xi_1(x) : x \in J(T) \setminus S\} > 0$. To this end let

$$\mathbb{N}(z) = \{n \ge p : T^n(z) \notin B(\omega(T), 2\eta)\}.$$

Suppose first that the set $\mathbb{N}(z)$ is infinite and let $\{n_k + p - 1\}_{k=1}^{\infty}$ be the sequence of consecutive elements of $\mathbb{N}(z)$. Then, in view of Lemma 3.1, there exists a unique holomorphic inverse branch $T_z^{-n_k} : B(T^{n_k}(z), 2\gamma) \to \overline{\mathbb{C}}$ such that

$T_z^{-n_k}(T^{n_k}(z)) = z$. Thus putting $\xi_1(z) = \min\{2\gamma, \xi\}$ we see that in this case the assumptions of Lemma 2.4(a) are satisfied.

Suppose now that $\mathbb{N}(z)$ is finite and let $i > \max \mathbb{N}(z)$. Since $z \notin S$, there exist $\alpha > 0$ and a unique holomorphic inverse branch $T_z^{-i} : B(T^i(z), \alpha) \to \overline{\mathbb{C}}$ such that $T_z^{-i}(T^i(z)) = z$. By (3.1), (3.3) and Lemma 2.2 for every $j \geq 1$ there exists a unique holomorphic inverse branch $T_i^{-j} : B(T^{i+j}(z), 2\theta) \to \overline{\mathbb{C}}$ such that $T_i^{-j}(T^{i+j}(z)) = T^i(z)$. In view of (3.3) and KDT for every j sufficiently large, $\text{diam}(T_i^{-j}(B(T^{i+j}(z), \theta))) < \alpha$. Therefore, for these j, the composition $T_z^{-i} \circ T_i^{-j}$ is well defined on $B(T^{i+j}(z), \theta)$ and so, setting $\xi_1(z) = \min\{\theta, \xi\}$, we conclude that the assumptions of Lemma 2.4(a) are again satisfied. Hence the remark that

$$\inf\{\xi_1(x) : x \in J(T) \setminus S\} = \min\{2\gamma, \theta, \xi\} > 0$$

finishes the proof. ∎

In a standard way one can apply the Besicovič Covering Theorem (as in [15], [16], [5], for example). Because of this, the previous lemma, $J(R) = J(T)$ and since any t-conformal measure for $R : J(R) \to J(R)$ is also t-conformal for $T : J(T) \to J(T)$, the following results hold.

COROLLARY 3.4: *Every two t-conformal measures m_1 and m_2 are equivalent on the set $J(R) \setminus S$. More precisely, there exists a constant $C_8 \geq 1$ (depending on m_1 and m_2) such that $C_8^{-1} m_1(A) \leq m_2(A) \leq C_8 m_1(A)$ for every Borel set $A \subset J(R) \setminus S$.*

Since (3.10) only holds for some radii, we cannot conclude at this point that a t-conformal measure and the t-dimensional Hausdorff measure are equivalent. However, we obtain the following preliminary result.

COROLLARY 3.5: *If m is a t-conformal measure for $R : J(R) \to J(R)$ and H_t is the t-dimensional Hausdorff measure on $J(R)$, then $H_t \ll m$. Moreover, the Radon–Nikodym derivative dH_t/dm is bounded.*

LEMMA 3.6: *If m_1 is a t_1-conformal measure, m_2 is a t_2-conformal measure with $t_1 < t_2$, then $m_2(J(R) \setminus S) = 0$.*

Now we shall prove the key lemma of the paper.

LEMMA 3.7: *If m is a nonatomic t-conformal measure for $R : J(R) \to J(R)$, then there exists a constant $C_9 \geq 1$ such that for every $z \in J(R)$ and every*

$0 < r \le 1$

$$C_9^{-1} \le \frac{m(B(z,r))}{r^t} \le C_9. \tag{3.11}$$

Proof: First of all note that $J(R) = J(T)$ and that m is also t-conformal for $T : J(T) \to J(T)$. Hence, in this proof we will always be dealing with the iteration T. For any $z \in J(T)$ let $C_9(z) \ge 1$ be the minimal constant such that (3.11) is satisfied. We need to prove that $C_9(z)$ is finite for every $z \in J(T)$ and, moreover, that $\sup\{C_9(z) : z \in J(T)\} < \infty$. The proof consists of several steps. Steps 1 and 2 deal with two special cases and are used later in the proof. In Step 3 it will be shown that (3.11) holds for $z \in \mathrm{Crit}(T^p)$, more precisely that the hypothesis (2.1) in Lemma 2.4(b) holds for any r, where the constants do not depend on z. If $z \notin \mathrm{Crit}(T^p)$ and r is given, we show in Steps 4 and 5 that at least one of the hypotheses of Lemma 2.4(b) is satisfied, where again the constants do not depend on z. Then Lemma 2.4(b) proves (3.11).

STEP 1: Let z be a point whose forward trajectory

$$\{T^n(z) : n \ge 0\} \subset B(\omega(T), 2\eta).$$

By (3.3) $z \notin S$, and hence Lemma 3.3, (3.9) applies. Also, by (3.1) and (3.3), Lemma 2.2 applies. In view of these two lemmas the hypotheses of Lemma 2.4(b) are satisfied with $\xi_2(z) = 2\theta$ and $C_5(z) = \|T'\|$. Moreover, formula (3.11) holds for z where the constant $C_9(z) < \infty$ does not depend on z.

STEP 2: Suppose that $z \in \mathrm{Crit}(T)$.

Then $z \in \mathrm{Crit}(T^p)$ and it follows from (3.2) that

$$\{T^p(z), T^{p+1}(z), \dots\} \subset B(\omega(T), 2\eta).$$

Note (see [9], for example) that there exist $0 < \rho < 1$ and conformal homeomorphisms $h_1 : B(0, 2\rho) \to \overline{\mathbb{C}}$ and $h_2 : B(0, 2\rho) \to \overline{\mathbb{C}}$ such that $h_1(0) = z$, $h_2(0) = T^p(z)$ and

$$T^p(h_1(x)) = h_2(x^q)$$

where $q \ge 2$ is the order of T^p at the point z.

Define finite Borel measures m_1 and m_2 on $B(0,\rho)$ by

$$m_1(A) = \int_{h_1(A)} |(h_1^{-1})'|^t \, dm, \qquad m_2(A) = \int_{h_2(A)} |(h_2^{-1})'|^t \, dm \tag{3.12}$$

for a measurable set $A \subset B(0,\rho)$. Using the chain rule and the properties of the conformal measure m it is easy to check that

$$m_2(A^q) = q^t \int_A |x^{q-1}|^t \, dm_1(x) \tag{3.13}$$

for every Borel set $A \subset B(0,\rho)$ on which the map $x \mapsto x^q$ is injective, where A^q denotes the image of the set A under the map $x \mapsto x^q$.

Note that Step 1 applies to $T^p(z)$, hence, since $|h_2'(0)| \neq 0, \infty$, it follows from KDT that

$$C_{10}^{-1} \le \frac{m_2(B(0,r))}{r^t} \le C_{10} \tag{3.14}$$

for every $0 < r < \rho$ and some constant $C_{10} \ge 1$.

Fix now any $0 < \nu < 1$. For every $0 < r < \rho$ and every $j = 0, 1, \ldots, q-1$ let

$$R(\nu r, r) = \{x : \nu r \le |x| < r\}$$

and

$$R_j(r) = \left\{be^{i\theta} : \sqrt[q]{\nu r} \le b < r \quad \text{and} \quad 2\pi j/q \le \theta < 2\pi(j+1)/q\right\}.$$

Note that the map $x \mapsto x^q$ is injective on $R_j(r)$, the sets $R_j(r)$ are mutually disjoint,

$$R(\sqrt[q]{\nu r, r}) = R_0(r) \cup \cdots \cup R_{q-1}(r) \quad \text{and} \quad (R_j(r))^q = R(\nu r^q, r^q).$$

Therefore by (3.13) and (3.14)

$$qC_{10}^{-1} r^{qt} \le q m_2(B(0,r^q)) = q^t \int_{B(0,r)} |x^{q-1}|^t \, dm_1(x) \le q^t r^{(q-1)t} m_1(B(0,r))$$

and

$$\begin{aligned} qC_{10} r^{qt} &\ge q m_2(R(\nu r^q, r^q)) = \sum_{j=0}^{q-1} q^t \int_{R_j(r)} |x^{q-1}|^t \, dm_1(x) \\ &\ge \sum_{j=0}^{q-1} (q^t \sqrt[q]{\nu r})^{(q-1)t} m_1(R_j(r)) = q^t \nu^{(q-1)t/q} r^{(q-1)t} m_1(R(\sqrt[q]{\nu r, r})). \end{aligned}$$

Hence

$$\frac{m_1(B(0,r))}{r^t} \ge C_{10}^{-1} q^{1-t} \tag{3.15}$$

and

$$m_1(R(\sqrt[q]{\nu r}, r)) \le C_{10} q^{1-t} \nu^{(q-1)t/q} r^t.$$

Since m has no atoms, also m_1 is nonatomic. Thus, setting

$$C_{11} = C_{10} q^{1-t} \nu^{(q-1)t/q},$$

we obtain that

$$\begin{aligned} m_1(B(0,r)) &= m_1\Big(\bigcup_{n=0}^{\infty} R((\sqrt[q]{\nu})^{n+1} r, (\sqrt[q]{\nu})^n r)\Big) \\ &= \sum_{n=0}^{\infty} m_1(R((\sqrt[q]{\nu})^{n+1} r, (\sqrt[q]{\nu})^n r)) \\ &\le \sum_{n=0}^{\infty} C_{11}((\sqrt[q]{\nu})^n r)^t \le C_{11} r^t \frac{1}{1-(\sqrt[q]{\nu})^t}. \end{aligned}$$

Therefore, as the set of critical points of T is finite, applying (3.15), (3.12) and KDT, we conclude that (3.11) is satisfied for any critical point z.

STEP 3: Let $z \in \mathrm{Crit}(T^p)$.

Let $0 \le i \le p-1$ be the smallest integer such that $T^i(z) \in \mathrm{Crit}(T)$. Hence there exist $\alpha > 0$ and a unique holomorphic inverse branch $T_z^{-i} : B(T^i(z), \alpha) \to \overline{\mathbb{C}}$ sending $T^i(z)$ to z. Therefore, as the set of critical points of T^p is finite, an application of Step 2 and KDT shows that condition (2.1) is satisfied for any r. In particular, (3.11) holds for any critical point of T^p.

STEP 4: Let $0 < r \le 1$ and $z \in J(T) \setminus \mathrm{Crit}(T^p)$ such that

$$r|(T^n)'(z)| \le 1$$

for every $n \ge 0$.

It follows from formula (3.9) of Lemma 3.3 that $z \in S$. Let $k \ge 0$ be the smallest integer such that $T^k(z) \in \mathrm{Crit}(T)$. Since $z \notin \mathrm{Crit}(T^p)$, there exists $l \ge 1$ such that $T^{p-1}(T^l(z)) = T^k(z)$. Moreover, since $T^k(z) \notin B(\omega(T), 2\eta)$, it follows from Lemma 3.1 that there exists a unique holomorphic inverse branch $T_z^{-l} : B(T^l(z), 2\gamma) \to \overline{\mathbb{C}}$ such that $T_z^{-l}(T^l(z)) = z$. But

$$(T^p)'(T^l(z)) = (T^{p-1})'(T^l(z)) \cdot T'(T^k(z)) = 0$$

and therefore an application of Step 3 to $T^l(z)$ and KDT shows that the hypothesis (2.1) holds for this r and this z.

STEP 5: Let $0 < r < \|T'\|^{-p+1}$ and $z \notin \mathrm{Crit}(T^p)$ such that

$$r|(T^n)'(z)| > 1 \tag{3.16}$$

for some $n \geq 1$.

Fix a minimal n satisfying (3.16). Since $r < \|T'\|^{-p+1}$, $n \geq p$ and

$$r|(T^n)'(z)| \leq \|T'\|. \tag{3.17}$$

CASE (a): Assume that $T^n(z) \notin B(\omega(T), 2\eta)$.

Hence, in view of Lemma 3.1, there exists a holomorphic inverse branch $T_z^{-n+p-1} : B(T^{n-p+1}(z), 2\gamma) \to \overline{\mathbb{C}}$ which sends the point $T^{n-p+1}(z)$ to the point z. In order to show the hypothesis of Lemma 2.4 (b), (2.3), for $n-p+1$, it suffices to note that $|(T^{p-1})'(T^{n-p+1}(z))| \leq \|T'\|^{p-1}$ and, as n is the smallest integer satisfying (3.17), $|(T^{p-1})'(T^{n-p+1}(z))| > 1$.

CASE (b): Assume that $T^n(z) \in B(\omega(T), 2\eta)$.

Let $k \geq 0$ be the smallest integer such that

$$\{T^k(z), T^{k+1}(z), \dots, T^n(z)\} \subset B(\omega(T), 2\eta).$$

CASE (b1): $k = 0$. In this case Step 1 applies.

CASE (b2): $k \geq 1$. Let

$$l = \max\{0, k-p\}.$$

Thus by (3.1), (3.3) and Lemma 2.2 (and (3.7)) there exists a unique holomorphic inverse branch $T_1^{-n+l+p} : B(T^n(z), 2\delta) \to B(T^{l+p}(z), 2\delta)$ such that $T_1^{-n+l+p}(T^n(z)) = T^{l+p}(z)$.

CASE (b2α): Suppose that

$$T^l(z) \notin B(\mathrm{Crit}(T^p), \varepsilon).$$

Then (see (3.7) and (3.8)) there exists a unique holomorphic inverse branch $T_2^{-p} : B(T^{l+p}(z), 2\delta) \to \overline{\mathbb{C}}$ such that $T_2^{-p}(T^{l+p}(z)) = T^l(z)$.

If $l = 0$ then the composition $T_2^{-p} \circ T_1^{-n+l+p} : B(T^n(z), 2\delta) \to \overline{\mathbb{C}}$ is a holomorphic inverse branch which sends $T^n(z)$ to z. Moreover, (2.3) holds because of (3.16) and (3.17).

If $l \geq 1$ then $l = k-p$ and $k-1 \geq p-1$. Moreover, since $T^{k-1}(z) \notin B(\omega(T), 2\eta)$, it follows from Lemma 3.1 that there exists a holomorphic inverse branch T_z^{-l} : $B(T^l(z), 2\gamma) \to \overline{\mathbb{C}}$ such that $T_z^{-l}(T^l(z)) = z$. By (3.8) the following composition $T_z^{-l} \circ T_2^{-p} \circ T_1^{-n+l+p} : B(T^n(z), 2\delta) \to \overline{\mathbb{C}}$ is well defined and sends $T^n(z)$ to z. Again, (2.3) follows from (3.16) and (3.17).

CASE (b2β): Suppose that

$$T^l(z) \in B(c, \varepsilon)$$

for some $c \in \mathrm{Crit}(T^p)$.

Put $\rho = |T^p(T^l(z)) - T^p(c)|$ and suppose that

$$\tau\rho|(T^{n-l-p})'(T^p(T^l(z)))| \geq K\delta,$$

where τ is given by Lemma 3.2. Then by KDT

$$T_1^{-n+l+p}(B(T^n(z), \delta)) \subset B(T^p(T^l(z)), \tau\rho).$$

Hence the composition $T_l^{-p} \circ T_1^{-n+l+p} : B(T^n(z), \delta) \to \overline{\mathbb{C}}$ sending $T^n(z)$ to $T^l(z)$ is well defined, where $T_l^{-p} : B(T^{l+p}(z), \tau\rho) \to B(T^l(z), \gamma)$ is a holomorphic inverse branch sending $T^{l+p}(z)$ to $T^l(z)$, which exists in view of Lemma 3.2.

If $l = 0$ we are done in view of (3.16) and (3.17).

If $l \geq 1$ then $l = k - p$. Since moreover $T^{k-1}(z) \notin B(\omega(T), 2\eta)$, it follows from Lemma 3.1 that there exists a holomorphic inverse branch $T_z^{-l} : B(T^l(z), 2\gamma) \to \overline{\mathbb{C}}$ such that $T_z^{-l}(T^l(z)) = z$. Hence the composition

$$T_z^{-l} \circ T_l^{-p} \circ T_1^{-n+l+p} : B(T^n(z), \delta) \to \overline{\mathbb{C}}$$

satisfies $T_z^{-l} \circ T_l^{-p} \circ T_1^{-n+p+l}(T^n(z)) = z$ and (2.3) follows again from (3.16) and (3.17).

Finally, let

$$\tau\rho|(T^{n-l-p})'(T^p(T^l(z)))| < K\delta.$$

Putting $\alpha = |T^l(z) - c|$, (3.4) implies then that

$$\tau A^{-1}\alpha^q|(T^{n-l-p})'(T^p(T^l(z)))| < K\delta. \tag{3.18}$$

But in view of (3.16) and (3.5)

$$r|(T^l)'(z)|A\alpha^{q-1}|(T^{n-l-p})'(T^{p+l+1}(z))| > 1$$

and, multiplying both sides of this inequality by α, using (3.18) and (3.7) we get

$$\alpha < \tau^{-1}K\delta A^2 r|(T^l)'(z)| < \tfrac{1}{3}\gamma r|(T^l)'(z)|,$$

which implies that

$$B(T^l(z), \tfrac{1}{3}\gamma r|(T^l)'(z)|) \subset B(c, \tfrac{2}{3}\gamma r|(T^l)'(z)|) \subset B(T^l(z), \gamma r|(T^l)'(z)|). \tag{3.19}$$

In view of (3.16) and the minimality of n

$$\gamma r|(T^l)'(z)| \le \gamma. \tag{3.20}$$

Therefore, if $l = 0$, applying the results of Step 3 for c gives

$$m(B(z, \tfrac{1}{3}\gamma r)) \le C_{12}(\tfrac{2}{3}\gamma)^t r^t$$

and

$$m(B(z, \gamma r)) \ge C_{12}^{-1}(\tfrac{2}{3}\gamma)^t r^t$$

with some constant $C_{12} \ge 1$. Thus (2.1) is satisfied in this case.

If $l \ge 1$ then $l = k - p$. Since moreover $T^{k-1}(z) \notin B(\omega(T), 2\eta)$, it follows from Lemma 3.1 that there exists a holomorphic inverse branch $T_z^{-l} : B(T^l(z), 2\gamma) \to \overline{\mathbb{C}}$ such that $T_z^{-l}(T^l(z)) = z$. Using (3.19), (3.20), the result of Step 3 for c and KDT we conclude that

$$m(B(z, \tfrac{1}{3}K^{-1}\gamma r)) \le C_{13}(\tfrac{2}{3}K\gamma)^t r^t$$

and

$$m(B(z, K\gamma r)) \ge C_{13}^{-1}(\tfrac{2}{3}K^{-1}\gamma)^t r^t$$

for some constant $C_{13} \ge 1$. Thus (2.1) is satisfied. ∎

Remark: Note that arguments used in Case (b2β) are similar to those contained in §2 "Telescope Lemma" of [13]. ∎

4. Hausdorff Measures for Subexpanding Rational Maps

The results in the previous section imply a few results on Hausdorff and conformal measures, as well as on absolutely continuous R-invariant measures, which we collect in this section.

In a standard way one can apply the Besicovič Covering Theorem (as in [15], [16], [5]) observing that $J(R) = J(T)$ and that any t-conformal measure for $R : J(R) \to J(R)$ is also t-conformal for $T : J(T) \to J(T)$. From this and Lemma 3.7 one obtains

THEOREM 4.1: *If m is a nonatomic t-conformal measure for $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ and if H_t is the t-dimensional Hausdorff measure on $J(R)$, then the measures m and H_t are equivalent with bounded Radon–Nikodym derivatives.*

In order to obtain information about Hausdorff measures H_t, we need to know that a nonatomic t-conformal measure exists. This is shown in the following.

THEOREM 4.2: *Let h denote the Hausdorff dimension of $J(R)$. Then there exists a nonatomic h-conformal measure m for $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Moreover, each other h-conformal measure is equivalent to m.*

Proof: In view of Corollary 3.4, in order to prove the theorem, it is enough to show the existence of a nonatomic h-conformal measure m. In [7] (Lemma 5.3, Theorem 2.3 and the proof of Lemma 5.4) we have constructed a sequence of probability measures $\{m_j\}_{j=1}^{\infty}$ on $J(R)$ and a sequence $\{h_j\}_{j=1}^{\infty}$ of positive reals with the following properties (among others):

$$m_j\left(\bigcup_{n=0}^{\infty} R^{-n}(\mathrm{Crit}(R))\right) = 0, \tag{4.1}$$

$$m_j(R(A)) \ge \int_A |R'|^{h_j}\, dm_j \tag{4.2}$$

for every set $A \subset J(R)$ such that $R|_A$ is injective and

$$s = \limsup_{j\to\infty} h_j \le h. \tag{4.3}$$

Let the measure m be defined as a weak accumulation point of the measures m_j. It is shown in [7] (Lemma 5.3, Corollary 5.7, Lemma 5.8 and Theorem 2.3) that for Misiurewicz's maps this measure is s-conformal. Therefore, in view of (4.3) and Corollary 3.5, $s = h$.

Let T be as in Section 3. Let $f(r) = \limsup_{j\to\infty} m_j(B(c,r))$, where $c \in \mathrm{Crit}(T^p)$ and $r > 0$. We shall show that

$$\lim_{r\to 0} f(r) = 0. \tag{4.4}$$

Indeed, it follows from (4.2) that $m_j(T(A)) \geq \int_A |T'|^{h_j}\, dm_j$ for every set $A \subset J(T) = J(R)$ such that $T|_A$ is injective. Therefore, as by (4.1) $m_j(c) = 0$ for every $j \geq 1$, in exactly the same way as in Steps 1–3 of the proof of Lemma 3.7 one shows that there exists $C_{14} \geq 1$ (not depending on j) such that

$$m_j(B(c,r)) \leq C_{14} r^{h_j}$$

for every $0 < r \leq 1$. This finishes the proof of (4.4). But (4.4) immediately implies that $m(\mathrm{Crit}(T^p)) = 0$. Thus $m(\bigcup_{n=0}^{\infty} R^{-n}(\mathrm{Crit}(R))) = 0$. Since, in view of (3.9) (the first claim in Lemma 3.3), no other point can be an atom of m, the proof of Theorem 4.2 is finished. ∎

Remark: It follows immediately from Corollary 3.5 that there are no t-conformal measures with $t < h$. From Lemma 3.6 and Theorem 4.2 one sees that there are no nonatomic t-conformal measures with $t > h$. However, it turns out that there are purely atomic, t-conformal measures for $t > h$ (necessarily concentrated on preimages of critical points). ∎

As an immediate consequence of Theorem 4.1, Theorem 4.2 and Lemma 3.7 we get the following.

THEOREM 4.3: *If h denotes the Hausdorff dimension of $J(R)$, then $0 < h < \infty$ and $h = \mathrm{bD}(J(R))$.*

The first part of Theorem 4.3 follows from Theorems 4.1 and 4.2. Lemma 3.7 was used to derive Theorems 4.1 and its proof to derive Theorem 4.2. Besides this, to get the equality $h = \mathrm{bD}(J(R))$ we need to apply Lemma 3.7 directly. Applying also Corollary 3.5 and the main result of [7] we obtain the following.

COROLLARY 4.4: *$h = \delta_0(R) = \delta(R) = s(R) = dD(J(R))$, where all these numbers have been defined in [7].*

We conclude this paper with some remarks about R-invariant probability measures which are absolutely continuous with respect to an h-conformal measure m. First we have the following.

PROPOSITION 4.5: *If m is an h-conformal measure for $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ then* $m(\omega(R)) = 0$.

Proof: First note that the classical Lebesgue Density Theorem holds for locally finite measures ν on $\mathbb{R}^d$ (an easy consequence of the Besicovič Covering Theorem), i.e. if $A \subset \mathbb{R}^d$ is a bounded Borel set with $\nu(A) < \infty$, then there exists a Borel set $A_0 \subset A$ with $\nu(A_0) = \nu(A)$ such that for $x \in A_0$

$$\lim_{r \to \infty} \frac{\nu(A \cap B(x,r))}{\nu(B(x,r))} = 1.$$

Using this, Lemma 3 in [10] also holds for the conformal measure m. Hence $m(\omega(R)) \in \{0,1\}$. Now, every forward invariant proper subset of $J(R)$ is nowhere dense in $J(R)$ (this follows from topological exactness of $R_{|J(R)}$).

Since a conformal measure is positive on open sets, it follows that $m(\omega(R)) = 0$. ∎

THEOREM 4.6: *Let h denote the Hausdorff dimension of $J(R)$. Then there exists a unique, ergodic, R-invariant probability measure μ, equivalent to the h-conformal measure m.*

Proof: Let

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} m \circ T^{-j}$$

and let μ denote a weak accumulation point of the sequence $\{\mu_n\}$. Then μ is R-invariant and it is left to show that μ is equivalent to m. Since m is h-conformal the proof of Theorem 3 in [10] also works in the present situation, once Lemma 6 in [10] is established. Moreover, the proof of Lemma 6 also carries over in the present situation, except formula (9) in [10]. However, the corresponding version of this formula for the measure m can be obtained with arguments used in Step 2 of Lemma 3.7, since m is nonatomic.

Similarly as in [10] one can prove that μ is metrically exact, in particular ergodic, and therefore unique. ∎

It follows from Theorem 4.1 and Theorem 4.2 that m is equivalent to H_h and an easy computation shows that the Radon–Nikodym derivative $\phi = dm/d\mathrm{H}_h$ satisfies the equality

$$\phi(R(z)) = \phi(z)$$

for m-almost every point $z \in J(T)$. Therefore, it follows from ergodicity of μ that ϕ is m-a.e. constant and we get the following.

THEOREM 4.7: *Let h denote the Hausdorff dimension of $J(R)$. Then the normalized h-dimensional Hausdorff measure H_h on $J(R)$ is the only h-conformal measure for $R : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$.*

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